

EASTON'S THEOREM IN THE PRESENCE OF WOODIN CARDINALS

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ABSTRACT. Under the assumption that δ is a Woodin cardinal and GCH holds, I show that if F is any class function from the regular cardinals to the cardinals such that (1) $\kappa < \text{cf}(F(\kappa))$, (2) $\kappa < \lambda$ implies $F(\kappa) \leq F(\lambda)$, and (3) δ is closed under F , then there is a cofinality-preserving forcing extension in which $2^\gamma = F(\gamma)$ for each regular cardinal $\gamma < \delta$, and in which δ remains Woodin. Unlike the analogous results for supercompact cardinals [Men76] and strong cardinals [FH08], there is no requirement that the function F be locally definable.

1. INTRODUCTION

Easton [Eas70] proved that the continuum function $\kappa \mapsto 2^\kappa$ on regular cardinals can be forced to behave in any way that is consistent with König's Theorem ($\kappa < \text{cf}(2^\kappa)$) and monotonicity ($\kappa < \lambda$ implies $2^\kappa \leq 2^\lambda$). I will say that F is an *Easton function* if F is a function from the class of regular cardinals to the class of cardinals satisfying (1) $\kappa < \text{cf}(F(\kappa))$ and (2) $\kappa < \lambda$ implies $F(\kappa) \leq F(\lambda)$. In the presence of large cardinals, there are additional restrictions on the possible behaviors of the continuum function on regular cardinals. For example, Scott proved that if GCH fails at a measurable cardinal κ , then GCH fails on a normal measure one subset of κ . It seems natural to ask:

Question 1. Given a large cardinal κ , what Easton functions can be forced to equal the continuum function on the regular cardinals, while preserving the large cardinal property of κ ?

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Menas [Men76] showed that if F is a “locally definable” Easton function (for a definition see [Men76, Theorem 18] or [FH08, Definition 3.16]), then there is a forcing extension $V[G]$ in which $2^\gamma = F(\gamma)$ for each regular cardinal γ and each supercompact cardinal in V remains supercompact in $V[G]$. In Menas’ proof, the local definability of F is needed to show that for an elementary embedding $j : V \rightarrow M$ witnessing the λ -supercompactness of κ , the functions F and $j(F)$ agree to an extent allowing one to lift j to $V[G]$. The developments in the literature addressing Question 1 in the case where κ is a measurable cardinal are more complicated. Woodin showed, using his method of modifying a generic filter, that if there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > \kappa^{++}$ and $M^\kappa \subseteq M$ then there is a forcing extension in which κ is measurable and GCH fails at κ (see [Cum10, Theorem 25.1] or [Jec03, Theorem 36.2]). In [FT08], Friedman and Thompson introduced the tuning fork method and argued that it provides a more streamlined proof of Woodin’s result. Friedman and Honzik [FH08] made use of the uniformity of the tuning fork method and provided an answer to Question 1 for measurable cardinals as well as for strong cardinals. Specifically, regarding strong cardinals, they proved that if F is any locally definable Easton function and GCH holds, then there is a cofinality preserving forcing extension $V[G]$ in which $2^\gamma = F(\gamma)$ for each regular cardinal γ and each strong cardinal in V remains strong in $V[G]$.

In this paper I prove the following theorem, which provides a complete answer to Question 1 for the case of Woodin cardinals (see Section 2.3 below for a definition and general discussion of Woodin cardinals).

Theorem 1. *Suppose GCH holds, $F : \text{REG} \rightarrow \text{CARD}$ is an Easton function, and δ is a Woodin cardinal closed under F . Then there is a cofinality-preserving forcing extension in which δ remains Woodin and $2^\gamma = F(\gamma)$ for each regular cardinal γ .*

The proof of Theorem 1 adapts the methods of [FH08] and [FT08] to a new case. Notice that in Theorem 1, there is no requirement stating that F must be *locally definable* as in the results of [Men76] and [FH08]. It is the property $j(A) \cap \gamma = A \cap \gamma$ in one of the characterizations of Woodin cardinals (see Lemma 10) that allows the removal of this additional requirement on F . Since a straight forward argument shows that $<\delta$ -closed forcing preserves the Woodinness of δ (see Lemma 12 below), the bulk of the work in proving Theorem 1 will be to show that the continuum function can be forced to agree with F below δ , while preserving the Woodinness of δ .

Let me remark here that as a corollary to the proof of Theorem 1, one has the following.

Corollary 2. *Suppose C is a class of Woodin cardinals and F is an Easton function such that δ is closed under F for each $\delta \in C$. Then there is a cofinality-preserving forcing extension in which δ remains Woodin and $2^\gamma = F(\gamma)$ for each regular cardinal γ .*

2. PRELIMINARIES FOR THE PROOF OF THEOREM 1

2.1. Lifting Embeddings. In what follows, I will be concerned with arguing that the Woodinness of a cardinal is preserved through forcing. This property is witnessed by elementary embeddings $j : M \rightarrow N$ between models of set theory. To show that such a large cardinal property is preserved to a forcing extension, say $V[G]$, one typically lifts the embedding to $j^* : M[G] \rightarrow N[j(G)]$ and argues that the lifted embedding witnesses the large cardinal property in $V[G]$. In this section, I will present some standard lemmas that are useful for lifting embeddings. For proofs of Lemmas 3 - 7, one may consult [Cum10] or [Cum92].

In what follows N and M are always assumed to be transitive models of ZFC. The following two lemmas are useful for building generic objects.

Lemma 3. *Suppose that $M^\lambda \subseteq M$ in V and there is in V an M -generic filter $H \subseteq \mathbb{Q}$ for some forcing $\mathbb{Q} \in M$. Then $M[H]^\lambda \subseteq M[H]$ in V .*

Lemma 4. *Suppose that $M \subseteq V$ is a model of ZFC, $M^{<\lambda} \subseteq M$ in V and \mathbb{P} is λ -c.c. If $G \subseteq \mathbb{P}$ is V -generic, then $M[G]^{<\lambda} \subseteq M[G]$ in $V[G]$.*

Suppose $j : M \rightarrow N$ is an embedding and $\mathbb{P} \in M$ a forcing notion. In order to lift j to $M[G]$ where G is M -generic for \mathbb{P} , one typically uses Lemmas 3 and 4 to build an N -generic filter H for $j(\mathbb{P})$ satisfying condition (1) in Lemma 5 below.

Lemma 5. *Let $j : M \rightarrow N$ be an elementary embedding between transitive models of ZFC. Let $\mathbb{P} \in M$ be a notion of forcing, let G be M -generic for \mathbb{P} and let H be N -generic for $j(\mathbb{P})$. Then the following are equivalent.*

- (1) $j''G \subseteq H$
- (2) *There exists an elementary embedding $j^* : M[G] \rightarrow N[H]$, such that $j^*(G) = H$ and $j^* \upharpoonright M = j$.*

The embedding j^* in condition (2) above is called a *lift* of j .

Suppose $j : V \rightarrow M$ is an elementary embedding. A set $S \in V$ is said to *generate j over V* if M is of the form

$$M = \{j(h)(s) \mid h : [A]^{<\omega} \rightarrow V, s \in [S]^{<\omega}, h \in V\}. \quad (2.1)$$

where $A \in V$ and $S \subseteq j(A)$. In this context, the elements of S are called seeds. For more on ‘seed theory’ and its applications, see [Ham97]. I will often make use of the following lemma which states that the above representation (2.1) of the target model of an elementary embedding remains valid after forcing.

Lemma 6. *If $j : V \rightarrow M$ is an elementary embedding generated over V by a set $S \in V$ then any lift of this embedding to a forcing extension $j^* : V[G] \rightarrow M[j^*(G)]$ is generated by S over $V[G]$ even if j^* is a class in some further forcing extension $N \supseteq V[G]$.*

The following standard lemma, which appears in [Cum92, Section 1.2], asserts that embeddings witnessed by extenders are preserved by highly distributive forcing.

Lemma 7. *If $j : V \rightarrow M$ is generated by $S \subseteq j(I)$, and $V[G]$ is obtained by $\leq |I|$ -distributive forcing, then j lifts uniquely to an embedding $j : V[G] \rightarrow M[j(G)]$.*

Proof. Suppose \mathbb{P} is $\leq |I|$ -distributive forcing and that G is V -generic for \mathbb{P} . By intersecting at most $|I|$ open dense subsets of \mathbb{P} , one may show that $j''G$ generates an M -generic filter on $j(\mathbb{P})$. \square

2.2. Iterations of Almost Homogeneous Forcing. In the course of proving Theorem 1, the next lemma will be used to show that a certain forcing iteration is almost homogeneous. Recall that a poset \mathbb{P} is *almost homogeneous* if for each pair of conditions, $p, q \in \mathbb{P}$, there is an automorphism $f \in \text{Aut}(\mathbb{P})$ such that $f(p)$ and q are compatible. If \mathbb{P} is an almost homogeneous forcing notion, a \mathbb{P} -name \dot{x} is called *symmetric* if for every automorphism $f \in \text{Aut}(\mathbb{P})$ one has $\Vdash_{\mathbb{P}} f(\dot{x}) = \dot{x}$, where $f(\dot{x})$ denotes the \mathbb{P} -name obtained from \dot{x} by recursively applying f .

Lemma 8. *Suppose $\mathbb{P}_\beta = \langle (\mathbb{P}_\alpha, \dot{Q}_\alpha) \mid \alpha < \beta \rangle$ is an Easton support iteration and that for each $\alpha < \beta$ one has $\Vdash_{\mathbb{P}_\alpha}$ “ \dot{Q}_α is almost homogeneous.” Suppose further that for each $\alpha < \beta$, one has that \dot{Q}_α is a symmetric \mathbb{P}_α -name; that is, for each automorphism $f \in \text{Aut}(\mathbb{P}_\alpha)$ one has $\Vdash_{\mathbb{P}_\alpha} f(\dot{Q}_\alpha) = \dot{Q}_\alpha$. Then the iteration \mathbb{P}_β is almost homogeneous.*

For a proof of Lemma 8 see [DF08, Lemma 4].

2.3. Some facts concerning Woodin cardinals. Woodin cardinals were originally formulated, by Woodin, for the purpose of reducing the large cardinal consistency strength needed for obtaining a model of the theory “every set of reals in $L(\mathbb{R})$ is Lebesgue measurable” (see the discussion around Theorem 32.9 in [Kan03]). Although part of the folklore, there has been little published, to the author’s knowledge, concerning the preservation of Woodin cardinals through forcing. For example, it is widely known that if δ is a Woodin cardinal, then the following forcing notions preserve this: (1) any forcing of size less than δ (see [HW00] for this result and more), (2) the canonical forcing to achieve GCH, and (3) any $<\delta$ -closed forcing (see Lemma 12 below).

I now give some further definitions and lemmas that will be used in the proof of Theorem 1. The following definition is due to Woodin.

Definition 9. A cardinal δ is called a *Woodin cardinal* if for every function $f : \delta \rightarrow \delta$ there is a $\kappa < \delta$ with $f''\kappa \subseteq \kappa$ and there is a $j : V \rightarrow M$ with critical point κ such that $V_{j(f)(\kappa)} \subseteq M$.

As it turns out, Woodin cardinals have another characterization which is more commonly used in practice. We present several versions of this characterization in the next lemma. First let me give a few definitions. Suppose $A \subseteq V_\delta$ and $\kappa < \delta$. One says that κ is γ -strong for A if there is a $j : V \rightarrow M$ with critical point κ such that $V_\gamma \subseteq M$, $j(\kappa) > \gamma$, and $j(A) \cap V_\gamma = A \cap V_\gamma$. By definition κ is $<\delta$ -strong for A if κ is γ -strong for A for each $\gamma < \delta$.

Lemma 10. *The following are equivalent.*

- (1) δ is a Woodin cardinal.
- (2) For every $A \subseteq V_\delta$ the following set is stationary.

$$\{\kappa < \delta \mid \kappa \text{ is } <\delta\text{-strong for } A\}$$

- (3) For every $A \subseteq V_\delta$ there is a $\kappa < \delta$ that is $<\delta$ -strong for A .
- (4) For every $A \subseteq \delta$ there is a $\kappa < \delta$ such that for any $\gamma < \delta$ there is a $j : V \rightarrow M$ with critical point κ such that $\gamma < j(\kappa)$ and $j(A) \cap \gamma = A \cap \gamma$.
- (5) For any pair of sets $A_0, A_1 \subseteq \delta$ there is a $\kappa < \delta$ such that for any $\gamma < \delta$ there is a $j : V \rightarrow M$ with critical point κ such that $\gamma < j(\kappa)$, $j(A_0) \cap \gamma = A_0 \cap \gamma$, and $j(A_1) \cap \gamma = A_1 \cap \gamma$.

For a proof that (1), (2), and (3) are equivalent one may see [Kan03, Theorem 26.14]. To see that (4) and (5) are equivalent to (3) one just needs to use standard coding techniques (details are worked out in [Cod12]).

If δ is a Woodin cardinal, then this is witnessed by embeddings as in Lemma 10(3). By considering a factor diagram, these embeddings can always be assumed to be extender embeddings (see [HW00]), meaning that the target of such an embedding, $j : V \rightarrow M$, is of the form

$$M = \{j(h)(a) \mid h : V_\kappa \rightarrow V, a \in V_\gamma, \text{ and } h \in V\}.$$

The following lemma will be required in our proof of Theorem 1.

Lemma 11. *Suppose κ is $<\delta$ -strong for $A \subseteq V_\delta$ where δ is a Woodin cardinal. There is a function $\ell : \kappa \rightarrow \kappa$ such that for any $\theta < \delta$ there is a $j : V \rightarrow M$ witnessing that κ is θ -strong for A such that $j(\ell)(\kappa) = \theta$.*

Proof. Define a function ℓ with domain κ as follows. If $\gamma < \kappa$ is not $<\delta$ -strong for A then define $\ell(\gamma)$ to be the least ordinal such that γ is not $\ell(\gamma)$ -strong for A . Otherwise define $\ell(\gamma) = 0$.

Let me show that $\ell(\gamma) < \kappa$ for each $\gamma < \kappa$. Suppose γ is not $<\delta$ -strong for A and that $\ell(\gamma) \geq \kappa$. I will show that since κ is $<\delta$ -strong for A it follows that γ is also $<\delta$ -strong for A , a contradiction. Choose $\theta < \delta$ and let $j : V \rightarrow M$ witness that κ is θ -strong for A . Since $\ell(\gamma) \geq \kappa$ it follows that γ is $<\kappa$ -strong for A . By elementarity $\gamma = j(\gamma)$ is $<j(\kappa)$ -strong for $j(A)$ in M . Thus γ is θ -strong for $j(A)$ in M . Let $i : M \rightarrow N$ witness this. Now let $j^* := i \circ j : V \rightarrow N$. It follows that γ is the critical point of j^* , that $j^*(\gamma) = i(j(\gamma)) = i(\gamma) > \theta$, and $j^*(A) \cap \theta = i(j(A)) \cap \theta = j(A) \cap \theta = A \cap \theta$. Hence γ is θ -strong for A . This implies that γ is $<\delta$ -strong for A , a contradiction. This shows that ℓ is a function from κ to κ .

Now fix $\theta < \delta$ and let $j : V \rightarrow M$ be an embedding witnessing that κ is θ -strong for A such that κ is not θ -strong for A in M . Such an embedding can be obtained by taking $j(\kappa)$ to be minimal. It follows that κ is β -strong for A in M for every $\beta < \theta$. Thus, $j(f)(\kappa) = \theta$. \square

The next widely known lemma is important for our proof of Theorem 1, because it easily implies that if δ is a Woodin cardinal, then one can force the continuum function to agree with any Easton function on the interval $[\delta, \infty)$.

Lemma 12. *If δ is a Woodin cardinal and \mathbb{P} is $<\delta$ -closed then δ remains Woodin after forcing with \mathbb{P} .¹*

Proof. For this proof, I will use the definition of Woodin cardinal as opposed to one of the characterizations given in Lemma 10. Let G be generic for \mathbb{P} and suppose $p \in G$ and $p \Vdash \dot{f} : \delta \rightarrow \delta$. Let D

¹I would like to thank Arthur Apter for an enlightening discussion concerning Lemma 12 and its proof.

be the set of conditions $q \leq p$ such that q forces there is a $\kappa < \delta$ such that $\dot{f}''\kappa \subseteq \kappa$ and there is a $j : V[\dot{G}] \rightarrow M[j(\dot{G})]$ with critical point κ and $(V_{j(\dot{f})(\kappa)})^{V[\dot{G}]} \subseteq M[j(\dot{G})]$. Note that the existence of the previous embedding is equivalent to the existence of an extender that has a first order definition. I will show that D is dense below p . Choose $r \leq p$ and use the $<\delta$ -closure of \mathbb{P} to find a descending sequence $\langle p_\alpha \mid \alpha < \delta \rangle$ of conditions below r such that p_α decides $\dot{f} \restriction (\alpha + 1)$ for each $\alpha < \delta$. Let $F : \delta \rightarrow \delta$ be the function in V determined by the sequence $\langle p_\alpha \mid \alpha < \delta \rangle$. By applying the Woodinness of δ in V to F find a $\kappa < \delta$ such that $F''\kappa \subseteq \kappa$ and there is a $j : V \rightarrow M$ with critical point κ and $V_{j(F)(\kappa)} \subseteq M$. In addition, by taking a factor embedding if necessary, one may assume without loss of generality that $M = \{j(h)(a) \mid h : V_\kappa \rightarrow V, a \in V_{j(F)(\kappa)}, \text{ and } h \in V\}$. Now choose $\alpha < \delta$ large enough so that p_α forces \dot{f} to agree with F up to and including at κ . Let H be V -generic for \mathbb{P} with $p_\alpha \in H$. Then $\dot{f}^H''\kappa \subseteq \kappa$. Since \mathbb{P} is $\leq \kappa$ -distributive, it follows by Lemma 7 that j lifts to $j : V[H] \rightarrow M[j(H)]$. By elementarity and the fact that $p_\alpha \in H$, it follows that $j(\dot{f}^H)(\kappa) = j(F)(\kappa)$. Since \mathbb{P} is $<\delta$ -closed, it follows that $(V_{j(F)(\kappa)})^{V[H]} = V_{j(F)(\kappa)}$. Thus, $(V_{j(\dot{f}^H)(\kappa)})^{V[H]} = (V_{j(F)(\kappa)})^{V[H]} = V_{j(F)(\kappa)} \subseteq M \subseteq M[j(H)]$. This shows that $p_\alpha \in D$ and thus that D is dense below p .

Now choose a condition $q \in G \cap D$ so that by the definition of D it follows that in $V[G]$ there is a $\kappa < \delta$ such that $f''\kappa \subseteq \kappa$ and there is a $j : V[G] \rightarrow M[j(G)]$ with critical point κ and $V[G]_{j(f)(\kappa)} \subseteq M[j(G)]$. \square

2.4. Sacks forcing on uncountable cardinals. Kanamori gave a definition for a version of Sacks forcing on uncountable cardinals in [Kan80]. In what follows, I will use a definition of Sacks forcing on inaccessible cardinals given by Friedman and Thompson in [FT08] (and used in [FH08]), which works particularly well for preserving large cardinals; for the reader's convenience, I will recall the definition and some basic properties of this forcing.

Suppose κ is an inaccessible cardinal. Then $p \subseteq 2^{<\kappa}$ is a *perfect κ -tree* if the following conditions hold.

- (1) If $s \in p$ and $t \in 2^{<\kappa}$ is an initial segment of s , then $t \in p$.
- (2) If $\langle s_\alpha \mid \alpha < \eta \rangle$ is a sequence of elements of p with $\eta < \kappa$ where $s_\alpha \subseteq s_\beta$ for $\alpha < \beta$, then $\bigcup_{\alpha < \eta} s_\alpha \in p$.
- (3) For each $s \in p$ there is a $t \in p$ with $s \subseteq t$ and $t \cap 0, t \cap 1 \in p$.
- (4) Let $\text{Split}(p) = \{s \in p \mid s \cap 0, s \cap 1 \in p\}$. Then for some unique closed unbounded set $C(p) \subseteq \kappa$, $\text{Split}(p) = \{s \in p \mid \text{length}(s) \in C(p)\}$.

Sacks forcing on κ is denoted by $\text{Sacks}(\kappa)$ and conditions in $\text{Sacks}(\kappa)$ are perfect κ -trees. For $p, q \in \text{Sacks}(\kappa)$, one says that p is stronger than q and writes $p \leq q$ if and only if $p \subseteq q$. For a condition $p \in \text{Sacks}(\kappa)$ let $\langle \alpha_i \mid i < \kappa \rangle$ be the increasing enumeration of $C(p)$. Let $\text{Split}_i(p) := \{s \in p \mid \text{length}(s) = \alpha_i\}$ denote the i^{th} splitting level of p . For $p, q \in \text{Sacks}(\kappa)$, define $p \leq_\beta q$ if and only if $p \leq q$ and $\text{Split}_i(p) = \text{Split}_i(q)$ for $i < \beta$. It is easy to verify that $\text{Sacks}(\kappa)$ is $<\kappa$ -closed and satisfies the κ^{++} -chain condition under GCH. By standard arguments, this implies that $\text{Sacks}(\kappa)$ preserves cardinals less than or equal to κ and greater than or equal to κ^{++} under GCH. Furthermore, as shown in [FT08], $\text{Sacks}(\kappa)$ satisfies the following fusion property. If $\langle p_\alpha \mid \alpha < \kappa \rangle$ is a decreasing sequence of conditions in $\text{Sacks}(\kappa)$ and for each $\alpha < \kappa$, $p_{\alpha+1} \leq_\alpha p_\alpha$, then the sequence has a lower bound in $\text{Sacks}(\kappa)$. The sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ is called a fusion sequence. This fusion property implies that $\text{Sacks}(\kappa)$ preserves κ^+ by the following straightforward argument. Suppose $p \Vdash \dot{f} : \check{\kappa} \rightarrow \check{\kappa}^+$. One can build a fusion sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ such that for each $\alpha < \kappa$, the condition $p_\alpha \in \text{Sacks}(\kappa)$ forces $\dot{f}(\check{\alpha})$ to equal the check name of an element of some set $A_\alpha = \{\beta_\xi \mid \xi < 2^\alpha\}$ where each β_ξ is less than κ^+ . By the fusion property, this sequence has a lower bound, call it r , and it follows that $r \Vdash \text{ran}(\dot{f}) \subseteq \bigcup_{\alpha < \kappa} A_\alpha$. Since $\bigcup_{\alpha < \kappa} A_\alpha$ has size at most κ , it follows that r forces $\text{ran}(\dot{f})$ to be bounded below κ^+ . The forcing $\text{Sacks}(\kappa)$ adds a single subset of κ given by a cofinal branch through $2^{<\kappa}$ and preserves cardinals under GCH.

Define $\text{Sacks}(\kappa, \lambda)$ to be the product forcing obtained by taking the product of λ -many copies of $\text{Sacks}(\kappa)$ with supports of size less than or equal to κ . Thus, a condition $\vec{p} \in \text{Sacks}(\kappa, \lambda)$ can be thought of as a function $\vec{p} : \lambda \rightarrow \text{Sacks}(\kappa)$ such that the set $\{\alpha < \lambda \mid \vec{p}(\alpha) \neq 2^{<\kappa}\}$ has size at most κ . The ordering on $\text{Sacks}(\kappa, \lambda)$ is given by the usual product ordering. It is easy to verify that $\text{Sacks}(\kappa, \lambda)$ is $<\kappa$ -closed and satisfies the κ^{++} -chain condition under GCH. Thus, assuming GCH, the poset $\text{Sacks}(\kappa, \lambda)$ preserves cardinals less than or equal to κ and greater than or equal to κ^{++} . To show that $\text{Sacks}(\kappa, \lambda)$ preserves κ^+ one may use the following generalized fusion property (see [FT08]). For $X \subseteq \lambda$ and $\vec{p}, \vec{q} \in \text{Sacks}(\kappa, \lambda)$ write $\vec{p} \leq_{\beta, X} \vec{q}$ if and only if $\vec{p} \leq \vec{q}$ and for each $\alpha \in X$, $\vec{p}(\alpha) \leq_\beta \vec{q}(\alpha)$. The generalized fusion property for $\text{Sacks}(\kappa, \lambda)$ asserts that if $\langle \vec{p}_\alpha \mid \alpha < \kappa \rangle$ is a descending sequence of conditions in $\text{Sacks}(\kappa, \lambda)$ and there is an increasing sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of subsets of λ , each of size less than κ , such that $\bigcup_{\alpha < \kappa} X_\alpha = \bigcup_{\alpha < \kappa} \text{supp}(\vec{p}_\alpha)$, and for each $\beta < \kappa$, $\vec{p}_{\beta+1} \leq_{\beta, X_\beta} \vec{p}_\beta$, then there is a lower bound of the sequence $\langle \vec{p}_\alpha \mid \alpha < \kappa \rangle$ in $\text{Sacks}(\kappa, \lambda)$. The above

generalized fusion property implies that κ^+ is preserved by the following argument. Suppose $\vec{p} \Vdash \dot{f} : \dot{\kappa} \rightarrow \dot{\kappa}^+$. One can build a fusion sequence $\langle \vec{p}_\alpha \mid \alpha < \kappa \rangle$ such that for each $\alpha < \kappa$, the condition \vec{p}_α forces $\dot{f}(\alpha)$ to belong to a subset of κ^+ of size $(2^\alpha)^\gamma$ for some $\gamma < \kappa$. A lower bound \vec{r} of this fusion sequence forces a bound on \dot{f} below κ^+ .

Since $\text{Sacks}(\kappa, \lambda)$ is not κ^+ -c.c. more than Lemma 4 will be required to see that $\text{Sacks}(\kappa, \lambda)$ preserves closure under κ sequences on inner models. For this reason will need the following.

Lemma 13. *Suppose $M \subseteq V$ is an inner model with $M^\kappa \subseteq M$ in V . If G is V -generic for $\text{Sacks}(\kappa, \lambda)$, then $M[G]^\kappa \subseteq M[G]$ in $V[G]$.*

Proof. Let me recall the proof given in [FT08, Lemma 3]. Let G be generic for $\text{Sacks}(\kappa, \lambda)$. Suppose X is a κ -sequence of ordinals in $V[G]$ and that this is forced by $p \in G$. Using generalized fusion, one can show that every $q \leq p$ can be extended to a condition r such that r forces that X can be determined from r and G . This implies that there is such an $r \in G$. Since r and G are both in $M[G]$, it follows that $X \in M[G]$. \square

Easton's Lemma states that if \mathbb{P} and \mathbb{Q} are forcing notions where \mathbb{P} is κ^+ -c.c. and \mathbb{Q} is $\leq \kappa$ -closed, then $\Vdash_{\mathbb{P}} \text{"}\mathbb{Q} \text{ is } \leq \kappa\text{-distributive."}$ The following lemma, which is analogous to Easton's Lemma, will be important for the proof of our main theorem.

Lemma 14. *Suppose \mathbb{P} is any $\leq \kappa$ -closed forcing and α is an ordinal. Then after forcing with $\text{Sacks}(\kappa, \alpha)$, \mathbb{P} remains $\leq \kappa$ -distributive.*

Proof. Suppose $p \in \text{Sacks}(\kappa, \lambda) \times \mathbb{P}$ forces that \dot{f} is a function with $\text{dom}(\dot{f}) = \kappa$. One can show, using generalized fusion in the first coordinate and closure in the second coordinate, that every condition q below p can be extended to a condition r which forces over $\text{Sacks}(\kappa, \lambda) \times \mathbb{P}$ that the values of \dot{f} can be determined from r and G , the generic for $\text{Sacks}(\kappa, \lambda)$. \square

For a more detailed proof of Lemma 14 see [FH08, Lemma 3.7].

3. PROOF OF THEOREM 1

Recall the statement of Theorem 1.

Theorem 3.1. *Suppose GCH holds, $F : \text{REG} \rightarrow \text{CARD}$ is an Easton function, and δ is a Woodin cardinal with $F''\delta \subseteq \delta$. Then there is a cofinality-preserving forcing extension in which δ remains Woodin and $2^\gamma = F(\gamma)$ for each regular cardinal γ .*

Proof.

Suppose δ is a Woodin cardinal and $F : \text{REG} \rightarrow \text{CARD}$ is an Easton function with $F''\delta \subseteq \delta$. For an ordinal α let $\bar{\alpha}$ denote the least closure point of F greater than α . For a regular cardinal γ , the notation $\text{Add}(\gamma, F(\gamma))$ denotes the poset for adding $F(\gamma)$ Cohen subsets to γ . The forcing is, the same iteration introduced in [FH08], that is, an Easton support iteration $\mathbb{P} = \langle (\mathbb{P}_\eta, \dot{\mathbb{Q}}_\eta) : \eta \in \text{ORD} \rangle$ of Easton support products defined as follows.

- (1) If η is an inaccessible closure point of F in $V^{\mathbb{P}_\eta}$, then $\dot{\mathbb{Q}}_\eta$ is a \mathbb{P}_η -name for the Easton support product

$$\text{Sacks}(\eta, F(\eta)) \times \prod_{\gamma \in (\eta, \bar{\eta}) \cap \text{REG}} \text{Add}(\gamma, F(\gamma))$$

as defined in $V^{\mathbb{P}_\eta}$ and $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta$

- (2) If η is a singular closure point of F in $V^{\mathbb{P}_\eta}$, then $\dot{\mathbb{Q}}_\eta$ is a \mathbb{P}_η -name for $\prod_{\gamma \in [\eta, \bar{\eta}) \cap \text{REG}} \text{Add}(\gamma, F(\gamma))$ as defined in $V^{\mathbb{P}_\eta}$ and $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta$.
- (3) Otherwise, if η is not a closure point of F , then $\dot{\mathbb{Q}}_\eta$ is a \mathbb{P}_η -name for trivial forcing and $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta$.

Let G be V -generic for \mathbb{P} . As in [FH08], it follows that cardinals are preserved (see [FH08, Lemma 3.6]) and that for each regular cardinal γ one has $2^\gamma = F(\gamma)$ (see [FH08, Theorem 3.8]).

Let me now discuss some notation that will be useful for factoring \mathbb{P} . If η is a closure point of F , then one can factor $\mathbb{P} \cong \mathbb{P}_\eta * \dot{\mathbb{P}}_{[\eta, \infty)}$ where \mathbb{P}_η denotes the iteration up to stage η and $\dot{\mathbb{P}}_{[\eta, \infty)}$ is a \mathbb{P}_η -name for the remaining stages. Thus G naturally factors as $G \cong G_\eta * G_{[\eta, \infty)}$. The stage η forcing in the iteration \mathbb{P} is \mathbb{Q}_η and I will write $\mathbb{Q}_\eta = \mathbb{Q}_{[\eta, \bar{\eta})}$ to emphasize the interval on which the stage η forcing has an effect. Let $H_{[\eta, \bar{\eta})}$ denote the $V[G_\eta]$ -generic for $\mathbb{Q}_{[\eta, \bar{\eta})}$ obtained from G . Let \mathbb{R}_γ denote a particular factor of the product forcing $\mathbb{Q}_{[\eta, \bar{\eta})}$ so that $\mathbb{Q}_{[\eta, \bar{\eta})} = \prod_{\gamma \in [\eta, \bar{\eta}) \cap \text{REG}} \mathbb{R}_\gamma$. In this situation let H_γ denote that $V[G_\eta]$ -generic for \mathbb{R}_γ obtained from G . In general, if $I \subseteq [\eta, \bar{\eta})$ then let $\mathbb{Q}_I = \prod_{\gamma \in I \cap \text{REG}} \mathbb{R}_\gamma$.

Since $\mathbb{P}_{[\delta, \infty)}$ is $<\delta$ -closed in $V^{\mathbb{P}_\delta}$, it follows by Lemma 12 that if δ is Woodin in $V^{\mathbb{P}_\delta}$ then δ remains Woodin in $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}_{[\delta, \infty)}}$. Thus it will suffice to show that δ remains Woodin in $V[G_\delta]$. Let me note here that by the previous statements, one could have defined the iteration above so that $\dot{\mathbb{P}}_{[\delta, \infty)}$ is simply a \mathbb{P}_δ -name for an Easton support product of Cohen forcing.

In what follows I will use the fact that since conditions in \mathbb{P}_δ have bounded support, one can view them as sequences of length less than δ . Indeed, by cutting off trivial coordinates, one can view a condition $p \in \mathbb{P}_\delta$ as being a condition in some initial segment of the poset.

I will show that property (3) in Lemma 10 holds in $V[G_\delta]$. Suppose $A \subseteq \delta$ with $A \in V[G_\delta]$ and let \dot{A} be a \mathbb{P}_δ -name for A . For each $\alpha < \delta$, let A_α be a maximal antichain of conditions in \mathbb{P}_δ that decide $\check{\alpha} \in \dot{A}$. Define a function $u : \delta \rightarrow \delta$ such that

$$u(\gamma) = \text{“the least ordinal } \beta \text{ such that for each } \alpha < \gamma, \\ \text{the antichain } A_\alpha \text{ is contained in } \mathbb{P}_\beta\text{”}$$

The value of $u(\gamma)$ indicates how much of the generic filter is required to correctly evaluate the name \dot{A} up to γ .

Now I will apply the Woodinness of δ in V . By an argument similar to that for Lemma 10(5), i.e. by coding the name $\dot{A} \subseteq V_\delta$, the Easton function $F \cap \delta \times \delta$, and the function $u \subseteq \delta \times \delta$, into a single subset of δ , it follows that there is a $\kappa < \delta$ that is $<\delta$ -strong for the name \dot{A} , the Easton function $F \restriction \delta$, and the function u . As an abbreviation, I will say that such a κ is $<\delta$ -strong for $\langle \dot{A}, F, u \rangle$. Since $C_F := \{\alpha < \delta \mid F''\alpha \subseteq \alpha\}$ is a closed unbounded subset of δ and since the set $S := \{\kappa < \delta \mid \kappa \text{ is } <\delta\text{-strong for } \langle \dot{A}, F, u \rangle\}$ is stationary, one may choose such a $\kappa \in C \cap S$. This is, of course, necessary since there is no hope of κ remaining measurable in $V[G_\delta]$ if κ is not a closure point of F .

Fix $\kappa < \delta$ such that κ is a closure point of F and κ is $<\delta$ -strong for $\langle \dot{A}, F, u \rangle$. Fix a function $\ell : \kappa \rightarrow \kappa$ as in Lemma 11. I will show that property (3) in Lemma 10 holds for this κ and the initially chosen $A \subseteq \delta$ in $V[G_\delta]$.

Since the inaccessible closure points of F are unbounded in δ , I may choose μ to be an inaccessible closure point of F with $F(\kappa) < \mu < \delta$. It will suffice to show that in $V[G_\delta]$ there is an embedding $j : V[G_\delta] \rightarrow M[j(G_\delta)]$ with critical point κ and $j(A) \cap \mu = A \cap \mu$. Now I will define a singular $\theta > \mu$ and lift an embedding that is θ -strong for $\langle \dot{A}, F, u \rangle$. I will also show that the lifted embedding satisfies $j(A) \cap \mu = A \cap \mu$. Using a singular degree of strength is advantageous since this will mean there will be no forcing over θ , and it will follow that the relevant tail forcing will be sufficiently closed. Let μ' be the least inaccessible closure point of u greater than μ . Define a sequence $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle$ by recursion as follows. Let γ_0 be the least inaccessible closure point of F greater than μ' . Assuming γ_α is defined where $\alpha < \kappa^+$, let $\gamma_{\alpha+1}$ be the least inaccessible closure point of F greater than γ_α . At limit

stages $\zeta < \kappa^+$, assuming $\langle \gamma_\alpha \mid \alpha < \zeta \rangle$ is defined, let γ_ζ be the least inaccessible closure point of F greater than $\sup\{\gamma_\alpha \mid \alpha < \zeta\}$. Now define $\theta := \sup\{\gamma_\alpha \mid \alpha < \kappa^+\}$. We have

$$\kappa < F(\kappa) < \mu < \mu' < \gamma_0 < \cdots < \gamma_\alpha < \cdots < \theta.$$

For emphasis, let me state the following explicitly.

- $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle$ is a discontinuous sequence of inaccessible closure points of F .
- $\theta = \sup\{\gamma_\alpha \mid \alpha < \kappa^+\}$
- $u''\mu' \subseteq \mu'$

By assumption on κ , there is a $j : V \rightarrow M$ with critical point κ such that the following hold.

- (1) $V_\theta \subseteq M$ & $\theta < j(\kappa)$
- (2) $j(\dot{A}) \cap \theta = \dot{A} \cap \theta$ & $j(F) \restriction \theta = F \restriction \theta$ & $j(u) \restriction \theta = u \restriction \theta$
- (3) $M = \{j(h)(s) \mid h : V_\kappa \rightarrow V, s \in V_\theta, h \in V\}$
- (4) $j(\ell)(\kappa) = \theta$ (using Lemma 11)

Since $j(F) \restriction \theta = F \restriction \theta$, the sequence $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle$ can be constructed in M from $j(F)$ just as it was constructed in V from F . This implies that

$$(5) \text{ cf}(\theta)^M = \kappa^+.$$

Property (5) will be important because it ensures that there is no forcing over θ in the iteration $j(\mathbb{P}_\delta)$.

3.1. Lifting j Through G_κ . In order to lift j to $V[G_\kappa]$, I will find an M -generic filter $j(G_\kappa)$ for $j(\mathbb{P}_\kappa)$ that satisfies $j''G_\kappa \subseteq j(G_\kappa)$. To do so, the length $j(\kappa)$ iteration $j(\mathbb{P}_\kappa)$ will be factored in M . Since $V_\theta \subseteq M$ it follows that $j(\mathbb{P}_\kappa) \cong \mathbb{P}_{\gamma_0} * \dot{\mathbb{P}}_{[\gamma_0, \theta)} * \dot{\mathbb{P}}_{[\theta, j(\kappa))}$ where $\dot{\mathbb{P}}_{[\gamma_0, \theta)}$ is a \mathbb{P}_{γ_0} -term for the iteration over the interval $[\gamma_0, \theta)$ as defined in $M^{\mathbb{P}_{\gamma_0}}$ and similarly $\dot{\mathbb{P}}_{[\theta, j(\kappa))}$ is a $\mathbb{P}_{\gamma_0} * \dot{\mathbb{P}}_{[\gamma_0, \theta)}$ -term for the tail of the iteration $j(\mathbb{P}_\kappa)$ as defined in $M^{\mathbb{P}_{\gamma_0} * \dot{\mathbb{P}}_{[\gamma_0, \theta)}}$. Since $V_\theta \subseteq M$, the iteration $j(\mathbb{P}_\kappa)$ agrees with \mathbb{P}_δ up to stage γ_0 . Thus it follows that G_{γ_0} is M -generic for \mathbb{P}_{γ_0} . Since θ is singular in V , conditions in $\mathbb{P}_{[\gamma_0, \theta)}$ are allowed to have unbounded support. Since M and V do not agree on the collection of unbounded subsets of θ , it follows by a density argument that $G_{[\gamma_0, \theta)}$ is not contained in $\tilde{\mathbb{P}}_{[\gamma_0, \theta)}$. Nonetheless, Lemmas 15 and 17 below will establish that there is an $M[G_{\gamma_0}]$ -generic filter, call it $\tilde{G}_{[\gamma_0, \theta)}$, in $V[G_{\gamma_0}][G_{[\gamma_0, \theta)}]$ for $\tilde{\mathbb{P}}_{[\gamma_0, \theta)}$. In Lemma 15, I will show that there is a condition $p_\infty \in \mathbb{P}_{[\gamma_0, \theta)}$ which forces all dense subsets of $\tilde{\mathbb{P}}_{[\gamma_0, \theta)}$ in $M[G_{\gamma_0}]$ to be met by $G_{[\gamma_0, \theta)}$. It might not be the case that $p_\infty \in G_{[\gamma_0, \theta)}$, but in

Lemma 17, I will show that p_∞ is in an automorphic image of $G_{[\gamma_0, \theta]}$, which I shall argue is good enough.

Let me note here that the proof of Lemma 15 resembles the construction of p_∞ in [FH08, Sublemma 3.12]. However, there is an important difference in that the forcing here, namely $\mathbb{P}_{[\gamma_0, \theta]}$, is an iteration involving generalize Sacks forcing, whereas in [FH08], the analagous forcing is a product of Cohen forcing.

Lemma 15. *There is a condition $p_\infty \in \mathbb{P}_{[\gamma_0, \theta]}$ such that if $G_{[\gamma_0, \theta]}^*$ is $V[G_{\gamma_0}]$ -generic for $\mathbb{P}_{[\gamma_0, \theta]}$ with $p_\infty \in G_{[\gamma_0, \theta]}^*$, then $G_{[\gamma_0, \theta]}^* \cap \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ is $M[G_{\gamma_0}]$ -generic for $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$.*

Proof. By our choice of θ , the sequence $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle$ is an increasing cofinal sequence of inaccessible closure points of F in θ . Recall the placement of the following ordinals.

$$\mu < \mu' < \gamma_0 < \gamma_1 < \cdots < \gamma_\alpha < \cdots < \theta$$

It follows that, in $M[G_{\gamma_0}]$, for each $\alpha < \kappa^+$,

$$\tilde{\mathbb{P}}_{[\gamma_0, \theta]} \cong \mathbb{P}_{[\gamma_0, \gamma_\alpha]} * \dot{\tilde{\mathbb{P}}}_{[\gamma_\alpha, \theta]}$$

where $\mathbb{P}_{[\gamma_0, \gamma_\alpha]}$ is γ_α^+ -c.c. in $V[G_{\gamma_0}]$ and $\dot{\tilde{\mathbb{P}}}_{[\gamma_\alpha, \theta]}$ is forced to be $< \gamma_\alpha$ -closed.

A few sublemmas will be required.

Sublemma 15.1. *Suppose $p_* = (r_*, \dot{q}_*) \in \mathbb{R} * \dot{\mathbb{Q}}$ and $D \subseteq \mathbb{R} * \dot{\mathbb{Q}}$ is open dense. Then there is an \mathbb{R} -name \dot{q}_D such that the following hold.*

- (1) $(r_*, \dot{q}_D) \leq (r_*, \dot{q}_*)$
- (2) $\bar{D} = \{r \leq r_* \mid (r, \dot{q}_D) \in D\}$ is open dense in \mathbb{R} below r_* .
- (3) $r_* \Vdash_{\mathbb{R}} \exists r \in \dot{G} (r, \dot{q}_D) \in D$

Proof. I will work below (r_*, \dot{q}_*) . Choose $(r_0, \dot{q}_0) \leq (r_*, \dot{q}_*)$ with $(r_0, \dot{q}_0) \in D$. Let $r'_0 \leq r$ with $r'_0 \perp r_0$. Now let $(r_1, \dot{q}_1) \leq (r'_0, \dot{q}_*)$ with $(r_1, \dot{q}_1) \in D$. Proceed by induction.

If α is a successor ordinal, say $\alpha = \beta + 1$, choose $r'_\beta \leq r_*$ with $r'_\beta \perp \{r_\xi \mid \xi \leq \beta\}$. Let $(r_{\beta+1}, \dot{q}_{\beta+1}) \in D$ with $(r_{\beta+1}, \dot{q}_{\beta+1}) \leq (r'_\beta, \dot{q}_*)$.

If α is a limit ordinal, suppose $\{r_\xi \mid \xi < \alpha\}$ is the antichain of \mathbb{R} constructed so far. Let $r''_\alpha \in \mathbb{R}$ be such that $r''_\alpha \perp \{r_\xi \mid \xi < \alpha\}$. Let $(r_\alpha, \dot{q}_\alpha) \in D$ with $(r_\alpha, \dot{q}_\alpha) \leq (r''_\alpha, \dot{q}_*)$.

The process terminates at some stage γ once $A := \{r_\xi \mid \xi < \gamma\}$ forms a maximal antichain of \mathbb{R} below r_* . Let \dot{q}_D be the \mathbb{R} -name obtained by mixing the names \dot{q}_ξ , defined above, over A . In other words, \dot{q}_D has the property that for each $\xi < \gamma$ the condition r_ξ forces $\dot{q}_D = \dot{q}_\xi$.

Let me show that (1) holds. Any generic for \mathbb{R} containing r_* will contain r_ξ for some $\xi < \gamma$. Since $r_\xi \Vdash \dot{q}_D = \dot{q}_\xi$ and $(r_\xi, \dot{q}_\xi) \leq (r_*, \dot{q}_*)$, it follows that $r_\xi \Vdash \dot{q}_D = \dot{q}_\xi \leq \dot{q}_*$. Hence $r_* \Vdash \dot{q}_D \leq \dot{q}_*$.

I will now show that (2) holds. Since D is open it easily follows that \bar{D} is open. Suppose $p \leq r_*$ with $p \in \mathbb{R}$. Since A is a maximal antichain of \mathbb{R} below r_* the condition p is compatible with some $r_\xi \in A$. Thus, let $s \in \mathbb{R}$ with $s \leq r_\xi$ and $s \leq p$. Since $(r_\xi, \dot{q}_\xi) \in D$ and D is open dense, to show that $s \in \bar{D}$ it will suffice to show that $(s, \dot{q}_D) \leq (r_\xi, \dot{q}_\xi)$. This easily follows since $s \leq r_\xi$ and $r_\xi \Vdash \dot{q}_D = \dot{q}_\xi$ imply that $s \Vdash \dot{q}_D \leq \dot{q}_\xi$. \square

Sublemma 15.2. *Suppose $q \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$. For all functions $h \in V$ with $\text{dom}(h) = V_\kappa$ and all $\beta < \theta$ there is a $p \leq q$ with $p \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ such that if $p \in G_{[\gamma_0, \theta]}^*$ is $V[G_{\gamma_0}]$ -generic for $\mathbb{P}_{[\gamma_0, \theta]}$, then $G_{[\gamma_0, \theta]}^*$ meets every dense subset of $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ of the form $j(h)(a)^{G_{\gamma_0}}$ where $a \in V_\beta$.*

Proof. Fix $q \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$, a function h , and β as in the statement of the sublemma. I will obtain the condition $p \leq q$ as a lower bound of a descending sequence of conditions in $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$. Since $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle$ is cofinal in θ , one may choose $\gamma_\alpha > |V_\beta|$. It follows that there is an enumeration $\vec{D} = \langle D_\xi^h \mid \xi < \zeta \rangle$, in $M[G_{\gamma_0}]$, of all dense subsets of $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ of the form $j(h)(a)^{G_{\gamma_0}}$ with $a \in V_\beta$. Clearly one has $\zeta \leq |V_\beta| < \gamma_\alpha$. Factor $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ as $\tilde{\mathbb{P}}_{[\gamma_0, \theta]} \cong \mathbb{P}_{[\gamma_0, \gamma_\alpha]} * \tilde{\mathbb{P}}_{[\gamma_\alpha, \theta]}$. In order to simplify notation, let me define $\mathbb{R} := \mathbb{P}_{[\gamma_0, \gamma_\alpha]}$ and $\dot{\mathbb{Q}} := \tilde{\mathbb{P}}_{[\gamma_\alpha, \theta]}$, so that $\tilde{\mathbb{P}}_{[\gamma_0, \theta]} \cong \mathbb{R} * \dot{\mathbb{Q}}$. Note that $\Vdash_{\mathbb{R}}$ “ $\dot{\mathbb{Q}}$ is $<\gamma_\alpha$ -closed.” Since $q \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]} \cong \mathbb{R} * \dot{\mathbb{Q}}$ one may write $q = (r_*, \dot{q}_*)$ where $r_* = q \restriction [\gamma_0, \gamma_\alpha] \in \mathbb{R}$ and \dot{q}_* denotes the \mathbb{R} -name, $q \restriction [\gamma_\alpha, \theta]$.

By the repeated application of Sublemma 15.1, and using the fact that $\Vdash_{\mathbb{R}}$ “ $\dot{\mathbb{Q}}$ is $<\gamma_\alpha$ -closed,” one may build a descending sequence of conditions $\langle (r_*, \dot{q}_\xi) \mid \xi \leq \zeta \rangle$ in $\mathbb{R} * \dot{\mathbb{Q}}$ such that for each $\xi \leq \zeta$, the set

$$\bar{D}_\xi^h := \{r \leq r_* \mid (r, \dot{q}_\xi) \in D_\xi^h\}$$

is dense below r_* in $\mathbb{R} = \mathbb{P}_{[\gamma_0, \gamma_\alpha]}$. Let $p := (r_*, \dot{q}_\zeta)$.

Suppose $p \in G_{[\gamma_0, \theta]}^*$ is $V[G_{\gamma_0}]$ -generic for $\mathbb{P}_{[\gamma_0, \theta]}$. Fix an $a \in V_\beta$ such that $j(h)(a)^{G_{\gamma_0}}$ is a dense subset of $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$. Since $j(h)(a)^{G_{\gamma_0}}$ must appear on the enumeration of dense sets we fixed above, there is a $\xi < \zeta$ such that $D_\xi^h = j(h)(a)^{G_{\gamma_0}}$. Since \bar{D}_ξ^h is dense below r_* in $\mathbb{R} = \mathbb{P}_{[\gamma_0, \gamma_\alpha]}$ there is a condition $r \in G_{[\gamma_0, \gamma_\alpha]}^* \cap \bar{D}_\xi^h$. By definition of \bar{D}_ξ^h , it follows that $(r, \dot{q}_\xi) \in D_\xi^h$. By padding r with $\mathbb{1}$'s, one sees that there is an \mathbb{R} -name \dot{b} such that $(r, \dot{b}) \in G_{[\gamma_0, \theta]}^*$. Since $p = (r_*, \dot{q}_\zeta)$ and (r, \dot{b}) are both in $G_{[\gamma_0, \theta]}^*$ they have a common extension $(r', \dot{q}') \in G_{[\gamma_0, \theta]}^*$. Since $(r', \dot{q}') \leq (r, \dot{q}_\zeta)$,

and since $r_* \Vdash \dot{q}_\zeta \leq \dot{q}_\xi$, it follows that $(r', \dot{q}') \leq (r, \dot{q}_\xi)$. Since $G_{[\gamma_0, \theta]}^*$ is a filter, one concludes that $(r, \dot{q}_\xi) \in G_{[\gamma_0, \theta]}^* \cap D_\xi^h$. \square

Continuing with the proof of Lemma 15, I will now use Sublemma 15.2 to construct the condition $p_\infty \in \mathbb{P}_{[\gamma_0, \theta]}$. Let $\langle f_\xi \mid \xi < \kappa^+ \rangle \in V$ be a sequence of functions with domain V_κ such that every dense subset of $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ in $M[G_{\gamma_0}]$ has a name of the form $j(f_\xi)(a)$ for some $\xi < \kappa^+$ and some $a \in V_\theta$. Let $w : \kappa^+ \rightarrow \kappa^+ \times \kappa^+$ be a bijection. It follows that $w \in M[G_{\gamma_0}]$ since $w \in V_\theta$. For each $\alpha < \kappa^+$ let $w(\alpha) = (w(\alpha)_0, w(\alpha)_1)$. The function w provides a well-ordering of pairs of the form (f_ξ, γ_α) . Notice that the well-ordering is not in $M[G_{\gamma_0}]$ since the sequence $\langle f_\xi \mid \xi < \kappa^+ \rangle$ is not in $M[G_{\gamma_0}]$. I will use this well-ordering of all pairs of the form (f_ξ, γ_α) of order type κ^+ to build a descending sequence of conditions $\langle p_\beta \mid \beta < \kappa^+ \rangle$ in $V[G_{\gamma_0}]$ with $p_\beta \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ such that if $p_\beta \in G_{[\gamma_0, \theta]}^*$ is $V[G_{\gamma_0}]$ -generic for $\mathbb{P}_{[\gamma_0, \theta]}$, then $G_{[\gamma_0, \theta]}^*$ meets $D_a^{f_\xi} = j(f_\xi)(a)_{G_{\gamma_0}}$ for each $a \in V_{\gamma_\alpha}$ where $w(\beta) = (\xi, \alpha)$. Since the above mentioned well-ordering will not be in $M[G_{\gamma_0}]$, I will need the next lemma to build the descending sequence.

Lemma 16. *The model $M[G_{\gamma_0}]$ is closed under κ -sequences in $V[G_{\gamma_0}]$.*

Proof. Since \mathbb{P}_κ is κ -c.c. in V (by [Jec03, Theorem 16.30]), it follows that $M[G_\kappa]^\kappa \subseteq M[G_\kappa]$ in $V[G_\kappa]$. By Lemma 13 it follows that $M[G_\kappa][H_\kappa]^\kappa \subseteq M[G_\kappa][H_\kappa]$ in $V[G_\kappa][H_\kappa]$. Since the remaining forcing $\mathbb{Q}_{[\kappa^+, \bar{\kappa}]} * \mathbb{P}_{[\bar{\kappa}, \gamma_0]}$ is $\leq \kappa$ -distributive in $V[G_\kappa][H_\kappa]$ (by Lemma 14) it follows that $M[G_{\gamma_0}]^\kappa \subseteq M[G_{\gamma_0}]$ in $V[G_{\gamma_0}]$. \square

I will now use the bijection $w : \kappa^+ \rightarrow \kappa^+ \times \kappa^+$ defined above to build the descending sequence. Let p_0 be the condition obtained by applying Sublemma 15.2 below the trivial condition to the function $h = f_\xi$ where $\xi = w(0)_0$ and to the ordinal $\beta = \gamma_\alpha$ where $\alpha = w(0)_1$. For successor stages, assume that $\langle p_\eta \mid \eta \leq \zeta \rangle$ has been constructed, where $\zeta < \kappa^+$. Let $p_{\zeta+1} \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ be obtained by applying Sublemma 15.2 below p_ζ to the function $h = f_\xi$ where $\xi = w(\zeta + 1)_0$ and to the ordinal $\beta = \gamma_\alpha$ where $\alpha = w(\zeta + 1)_1$. At limit stages $\zeta < \kappa^+$, assume $\langle p_\eta \mid \eta < \zeta \rangle$ has been constructed. The fact that $M[G_{\gamma_0}]^\kappa \subseteq M[G_{\gamma_0}]$ implies that the sequence $\langle p_\eta \mid \eta < \zeta \rangle$ is in $M[G_{\gamma_0}]$ since it has been constructed from an initial segment of $\langle f_\xi \mid \xi < \kappa^+ \rangle$ and from $\langle \gamma_\alpha \mid \alpha < \kappa^+ \rangle \in M[G_{\gamma_0}]$. Since $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ is $< \gamma_0$ -closed in $M[G_{\gamma_0}]$, one may let $p'_\zeta \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]}$ be a lower bound of $\langle p_\beta \mid \beta < \zeta \rangle$. Now let p_ζ be obtained by applying Sublemma 15.2 below p'_ζ to the function f_ξ where $\xi = w(\zeta)_0$ and the ordinal $\beta = \gamma_\alpha$ where $\alpha = w(\zeta)_1$.

This defines the sequence $\langle p_\eta \mid \eta < \kappa^+ \rangle$ in $V[G_{\gamma_0}]$ where $p_\eta \in \tilde{\mathbb{P}}_{[\gamma_0, \theta]} \subseteq \mathbb{P}_{[\gamma_0, \theta]}$ for each $\eta < \kappa^+$. Let $p_\infty \in \mathbb{P}_{[\gamma_0, \theta]}$ be a lower bound of $\langle p_\eta \mid \eta < \kappa^+ \rangle$.

Suppose $p_\infty \in G_{[\gamma_0, \theta]}^*$ is $V[G_{\gamma_0}]$ -generic for $\mathbb{P}_{[\gamma_0, \theta]}$. Suppose $D \in M[G_{\gamma_0}]$ is a dense subset of $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$. Then $D = D_a^{f_\xi} = j(f_\xi)(a)^{G_{\gamma_0}}$ for some $\xi < \kappa^+$ and where $a \in V_{\gamma_\alpha}$ for some $\alpha < \kappa^+$. Let $\zeta < \kappa^+$ with $w(\zeta) = (w(\zeta)_0, w(\zeta)_1) = (\xi, \alpha)$. Since $p_\infty \leq p_\zeta$, it follows that $p_\zeta \in G_{[\gamma_0, \theta]}^*$ and hence, $G_{[\gamma_0, \theta]}^*$ meets $D_a^{f_\xi}$, by Sublemma 15.2.

This concludes the proof of Lemma 15. \square

I will now show that there is an automorphic image of $G_{[\gamma_0, \theta]}$ containing p_∞ .

Lemma 17. *Suppose $c \in \mathbb{P}_{[\gamma_0, \theta]}$. There is an automorphism $\pi : \mathbb{P}_{[\gamma_0, \theta]} \rightarrow \mathbb{P}_{[\gamma_0, \theta]}$ in $V[G_{\gamma_0}]$ such that $c \in \pi'' G_{[\gamma_0, \theta]}$.*

Proof. Working in $V[G_{\gamma_0}]$, I claim each stage in the iteration $\mathbb{P}_{[\gamma_0, \theta]}$ is forced to be homogeneous over the previous stages. Let me argue that the Easton support product

$$\mathbb{Q}_\eta := \text{Sacks}(\eta, F(\eta)) \times \prod_{\gamma \in (\eta, \bar{\eta}) \cap \text{REG}} \text{Add}(\gamma, F(\gamma))$$

as defined in $V[G'_\eta]$ is almost homogeneous in $V[G'_\eta]$ where G'_η is any generic for \mathbb{P}_η . It will suffice to argue that each factor in the product \mathbb{Q}_η is almost homogeneous since automorphisms of each coordinate can be combined to give an automorphism of the product. Clearly, each factor of Cohen forcing $\text{Add}(\gamma, F(\gamma))$ is almost homogeneous. If $p, q \in \text{Sacks}(\eta, F(\eta))$ let f be an automorphism of $\text{Sacks}(\eta, F(\eta))$ such that the support of $f(p)$ is disjoint from the support of q . Then $f(p)$ is compatible with q .

By Lemma 8, to show that $\mathbb{P}_{[\gamma_0, \theta]}$ is almost homogeneous in $V[G_{\gamma_0}]$, it will suffice to show that at each stage $\eta \in [\gamma_0, \theta)$, the name $\dot{\mathbb{Q}}_\eta$ is a symmetric \mathbb{P}_η -name for the stage η forcing. Let me fix an automorphism f of $\mathbb{P}_{[\gamma_0, \eta]}$ and argue that $\Vdash_{\mathbb{P}_{[\gamma_0, \eta]}} f(\dot{\mathbb{Q}}_\eta) = \dot{\mathbb{Q}}_\eta$. There is a first order formula $\varphi(x_0, \dots, x_n)$ such that $\Vdash_{\mathbb{P}_{[\gamma_0, \eta]}} \text{“}\forall x [x \in \dot{\mathbb{Q}}_\eta \text{ if and only if } \varphi(x, \check{a}_1, \dots, \check{a}_n)]\text{”}$ where a_1, \dots, a_n are elements of the ground model $V[G_{\gamma_0}]$. Applying f to the previous statement one obtains $\Vdash_{\mathbb{P}_{[\gamma_0, \eta]}} \text{“}\forall x [x \in f(\dot{\mathbb{Q}}_\eta) \text{ if and only if } \varphi(x, \check{a}_1, \dots, \check{a}_n)]\text{”}$. Thus, if \dot{x} is a $\mathbb{P}_{[\gamma_0, \eta]}$ -name in $V[G_{\gamma_0}]$, it follows that

$$\Vdash_{\mathbb{P}_\eta} \dot{x} \in \dot{\mathbb{Q}}_\eta \longleftrightarrow \varphi(\dot{x}, \check{a}_1, \dots, \check{a}_n) \longleftrightarrow \dot{x} \in f(\dot{\mathbb{Q}}_\eta)$$

and hence $\Vdash_{\mathbb{P}_\eta} \dot{Q}_\eta = f(\dot{Q}_\eta)$. Applying Lemma 8, one concludes that $\mathbb{P}_{[\gamma_0, \theta]}$ is almost homogeneous in $V[G_{\gamma_0}]$.

Now it follows by an easy density argument that every condition $p \in \mathbb{P}_{[\gamma_0, \theta]}$ can be extended to a condition $q \leq p$ such that there is an $f \in \text{Aut}(\mathbb{P}_{[\gamma_0, \theta]})$ with $f(q) \leq c$. Therefore, by the genericity of $G_{[\gamma_0, \theta]}$, there is such a $q \in G_{[\gamma_0, \theta]}$ with such an $f \in \text{Aut}(\mathbb{P}_{[\gamma_0, \theta]})$. Let $\pi := f$. Since $\pi''G_{[\gamma_0, \theta]}$ is a filter and $\pi(q) \leq c$, it follows that $c \in \pi''G_{[\gamma_0, \theta]}$. \square

As discussed above, one may use Lemmas 15 and 17 to obtain $\tilde{G}_{[\gamma_0, \theta]} \in V[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$, an $M[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$ -generic for $\tilde{\mathbb{P}}_{[\gamma_0, \theta]}$.

To finish lifting j through $j(\mathbb{P}_\kappa) \cong \mathbb{P}_{\gamma_0} * \tilde{\mathbb{P}}_{[\gamma_0, \theta]} * \tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$, I will build an $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$ -generic for $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ in $V[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$. The following lemma will be required.

Lemma 18. *$M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$ is closed under κ -sequences in $V[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$.*

Proof. Since \mathbb{P}_κ is κ -c.c., it follows by Lemma 4 that $M[G_\kappa]$ is closed under κ -sequences in $V[G_\kappa]$. It is shown in [FH08, Lemma 3.14] and [FT08, Lemma 3], using a fusion argument, that $M[G_\kappa][H_\kappa]$ is closed under κ -sequences in $V[G_\kappa][H_\kappa]$. It will suffice to show that $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$ has every κ -sequence of ordinals in $V[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$. Suppose \vec{x} is a κ -sequence of ordinals in $V[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$. Then since $\mathbb{Q}_{[\kappa^+, \bar{\kappa}]} * \mathbb{P}_{[\bar{\kappa}, \theta]}$ is $\leq \kappa$ -distributive in $V[G_\kappa][H_\kappa]$, it follows that $\vec{x} \in V[G_\kappa][H_\kappa]$. Thus $\vec{x} \in M[G_\kappa][H_\kappa] \subseteq M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$. \square

Suppose D is a dense subset of $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ in $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$. Let $\dot{D} \in M$ be a nice \mathbb{P}_θ -name for D . Let h be a function in V with $\text{dom}(h) = V_\kappa$ and $s \in V_\theta$ with $\dot{D} = j(h)(s)$. Without loss of generality, assume that $\text{ran}(h)$ is contained in the set of nice names for dense subsets of a particular tail of \mathbb{P} . Since θ is singular, $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ is $\leq \theta$ -closed in $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$. The collection $\mathcal{D} := \{j(h)(s)_{G_{\gamma_0} * \tilde{G}_{[\gamma_0, \theta]}} \mid s \in V_\theta\}$ is in $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$. Since θ is a \beth -fixed point, there are at most θ dense subsets of $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ in \mathcal{D} . Thus, there is a single condition in $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ that meets every dense set in \mathcal{D} . Since there are at most κ^+ functions from V_κ to nice names for dense subsets of a tail of \mathbb{P}_κ , and since every dense subset of $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ has a name in M which is represented by such a function, the above procedure can be iterated to obtain a descending κ^+ -sequence of conditions in $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ meeting every dense subset of $\tilde{\mathbb{P}}_{[\theta, j(\kappa)]}$ in $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$. Let \tilde{G}_{tail} be the $M[G_{\gamma_0}][\tilde{G}_{[\gamma_0, \theta]}]$ -generic filter for \mathbb{P}_{tail} generated by this sequence.

Now let $j(G_\kappa) := G_{\gamma_0} * \tilde{G}_{[\gamma_0, \theta]} * \tilde{G}_{tail}$ and note that $j''G_\kappa \subseteq j(G_\kappa)$ since conditions in G_κ have support bounded below the critical point of j . Hence by Lemma 5, the embedding lifts to

$$j : V[G_\kappa] \rightarrow M[j(G_\kappa)]$$

in $V[G_{\gamma_0}][G_{[\gamma_0, \theta]}]$.

3.2. Lifting j Through $\text{Sacks}(\kappa, F(\kappa))$. It remains to show that the embedding lifts further through the forcing $\mathbb{P}_{[\kappa, \lambda]}$. I will now argue that j lifts through $\mathbb{R}_\kappa = \text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$, the first factor of the stage κ forcing. I will use the tuning fork method of [FT08] to construct an $M[j(G_\kappa)]$ -generic for $j(\mathbb{R}_\kappa) = \text{Sacks}(j(\kappa), j(F(\kappa)))^{M[j(G_\kappa)]}$ in $V[G_\kappa][H_\kappa]$ that satisfies the lifting criterion in Lemma 5. Say that $t \subseteq 2^{<j(\kappa)}$ is a *tuning fork that splits at κ* if and only if $t = t^0 \cup t^1$ where t^0 and t^1 are two distinct cofinal branches of $2^{<j(\kappa)}$ such that $t^0 \cap \kappa = t^1 \cap \kappa$, $t^0(\kappa) = 0$, and $t^1(\kappa) = 1$. For $\alpha < j(F(\kappa))$ let

$$t_\alpha := \bigcap \{j(p)(\alpha) \mid p \in H_\kappa\}.$$

The next lemma is key.

Lemma 19. *If $\alpha \in j''F(\alpha)$ then t_α is a tuning fork that splits at κ . Otherwise, if $\alpha < j(F(\kappa))$ is not in the range of j , then t_α is a cofinal branch through $2^{<j(\kappa)}$.*

Proof. The following proof follows [FT08] closely, except that here Lemma 11 is required. Working in $V[G_\kappa]$, let

$$X := \bigcap \{j(C) \mid C \subseteq \kappa \text{ is club and } C \in V\}.$$

First let me show that $X = \{\kappa\}$. If $\alpha < \kappa$ then clearly $\alpha \notin X$ since there is a closed unbounded subset C of κ whose least element is greater than α , and thus $\alpha \notin j(C)$. Since the limit cardinals below κ form a closed unbounded subset of κ it follows that any element of X must be a limit cardinal in $M[j(G_\kappa)]$ which is greater than or equal to κ . Suppose $\lambda < j(\kappa)$ is a limit cardinal and $\lambda > \theta$. Then $\lambda = j(h)(a)$ for some function $h : V_\kappa \rightarrow \kappa$ in $V[G_\kappa]$ and some $a \in V_\theta$. Let $C_h := \{\gamma < \kappa \mid \gamma \text{ is a limit cardinal and } h''V_\gamma \subseteq \gamma\}$. Then C_h is a closed unbounded subset of κ and $\lambda \notin j(C_h)$ since $\lambda > \theta$ and $j(h)''V_\lambda \not\subseteq \lambda$. Now suppose $\kappa < \lambda \leq \theta$. Above, the function ℓ is chosen using Lemma 11 so that $\ell : \kappa \rightarrow \kappa$ and $j(\ell)(\kappa) = \theta$. Then $C_\ell := \{\gamma < \kappa \mid \ell''\gamma \subseteq \gamma\}$ is a closed unbounded subset of κ in $V[G_\kappa]$ and $\lambda \notin j(C_\ell)$ since $\theta \in j(\ell)''\lambda$ and this implies $j(\ell)''\lambda \not\subseteq \lambda$. This shows that $X \subseteq \{\kappa\}$. Clearly $\kappa \in X$ since for each closed unbounded $C \subseteq \kappa$ in $V[G_\kappa]$, $j(C) \cap \kappa = C$.

The rest of the proof is exactly as in [FT08] and [FH08].

Let C be any closed unbounded subset of κ in $V[G_\kappa]$. Choose $\alpha < j(F(\kappa))$ and write $\alpha = j(f)(a)$ where $f : V_\kappa \rightarrow F(\kappa)$ and $a \in V_\theta$. It is easy to show that the following set is dense in $\text{Sacks}(\kappa, F(\kappa))$.

$$D_C = \{p \in \text{Sacks}(\kappa, F(\kappa)) \mid \xi \in \text{ran}(f) \implies C(p(\xi)) \subseteq C\}$$

Thus there is a $p \in H_\kappa \cap D_C$ with $C(j(p)(\alpha)) \subseteq j(C)$. Since C was an arbitrary closed unbounded subset of κ , this, together with the fact that $X = \{\kappa\}$, implies that t_α can only possibly split at κ . If $\alpha \in \text{ran}(j)$ then since κ is a limit point of $j(C)$ for every closed unbounded $C \subseteq \kappa$ in $V[G_\kappa]$, it follows that t_α splits at κ and is a tuning fork.

If $\alpha \notin \text{ran}(j)$ then $\text{ran}(f)$ must have size κ since otherwise $\alpha \in j(\text{ran}(f)) = j'' \text{ran}(f)$. Let $\langle \bar{\alpha}_i \mid i < \kappa \rangle$ enumerate $\text{ran}(f)$. Then $j(\langle \bar{\alpha}_i \mid i < \kappa \rangle) = \langle \alpha_i \mid i < j(\kappa) \rangle$ in an enumeration of $\text{ran}(j(f))$. It is easy to see that the set of conditions $p \in \text{Sacks}(\kappa, F(\kappa))$ such that for each $i < \kappa$, the least splitting level of $p(\bar{\alpha}_i)$ is above level i is dense. Thus there is a $p \in H_\kappa$ such that for each $i < j(\kappa)$ the least splitting level of $j(p)(\alpha_i)$ is beyond level i . Since $\alpha \notin \text{ran}(j)$ it follows that $\alpha = \alpha_i$ for some $i \in [\kappa, j(\kappa))$. It follows that the first splitting level of $j(p)(\alpha)$ is above κ . Thus, t_α is a cofinal branch. \square

Each t_α generates an $M[j(G_\kappa)]$ -generic filter for $j(\mathbb{R}_\kappa)$ as follows. For $\alpha \in j'' F(\kappa)$, let t_α^0 and t_α^1 be the left-most and right-most branches of t_α respectively; that is, for $k \in \{0, 1\}$ let

$$t_\alpha^k := \{s \in t_\alpha \mid \kappa \in \text{dom}(s) \implies s(\kappa) = k\}.$$

For $\alpha < j(F(\kappa))$ not in the range of j , let $t_\alpha^0 := t_\alpha$ be the cofinal branch in Lemma 19. Let

$$g := \{\tilde{p} \in j(\mathbb{R}_\kappa) \mid \forall \alpha < j(F(\kappa)) \ t_\alpha^0 \subseteq \tilde{p}(\alpha)\}.$$

It is easy to check that $j'' H_\kappa \subseteq g$, so to show that j lifts through \mathbb{R}_κ it remains to show that g is $M[j(G_\kappa)]$ -generic for $j(\mathbb{R}_\kappa)$. For this the following two definitions will be used, both of which are given in [FT08]. Suppose $p \in \text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$, $S \subseteq F(\kappa)$ with $|S|^{V[G_\kappa]} < \kappa$. Friedman and Thompson say that an (S, α) -*thinning* of p is an extension of p obtained by thinning each $p(\xi)$ for $\xi \in S$ to the subtree

$$p(\xi) \upharpoonright s_\xi := \{s \in p(\xi) \mid s_\xi \subseteq s \text{ or } s \subseteq s_\xi\}$$

where s_ξ is some particular node of $p(\xi)$ on the α -th splitting level of $p(\xi)$. A condition $p \in \text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$ is said to *reduce* a dense subset D of $\text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$ if and only if for some $S \subseteq F(\kappa)$ of size less than κ in $V[G_\kappa]$, any (S, α) -thinning of p meets D .

Let me now argue that g is $M[j(G_\kappa)]$ -generic for the poset $j(\mathbb{R}_\kappa) = \text{Sacks}(j(\kappa), j(F(\kappa)))^{M[j(G_\kappa)]}$. Suppose D is a dense subset of $j(\mathbb{R}_\kappa)$ in

the model $M[j(G_\kappa)]$. Then by Lemma 6 one can write $D = j(h)(a)$ where $h \in V[G_\kappa]$ is a function from V_κ to the collection of dense subsets of $\text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$ and $a \in V_\theta$. Let $\langle D_\beta \mid \beta < \kappa \rangle \in V[G_\kappa]$ enumerate the range of h . One may show, as in [FT08] that any condition $p \in \text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$ can be extended to $q \leq p$ which reduces each D_β for $\beta < \kappa$. This implies that the following is a dense subset of $\text{Sacks}(\kappa, F(\kappa))^{V[G_\kappa]}$.

$$D' := \{p \in \mathbb{R}_\kappa \mid p \text{ reduces each } D_\beta \text{ for } \beta < \kappa\}$$

Thus one may choose a condition $p \in H \cap D'$. By elementarity $j(p)$ reduces each dense subset of $j(\mathbb{R}_\kappa)$ in the range of $j(h)$; in particular, $j(p)$ reduces $D = j(h)(a)$. Thus it follows that there is an $S \subseteq j(F(\kappa))$ of size less than $j(\kappa)$ and an $\alpha < j(\kappa)$ such that any (S, α) -thinning of $j(p)$ meets D . For each $\xi \in S$ let $\tilde{q}(\xi)$ be the thinning of $j(p)(\xi)$ obtained by choosing an initial segment of t_ξ^0 on the α -th splitting level of $j(p)(\xi)$. For $\xi \in j(F(\kappa)) \setminus S$ let $\tilde{q}(\xi) := j(p)(\xi)$. The fact that \tilde{q} is a condition in $j(\mathbb{R}_\kappa)$ will follow from the next lemma, which appears in [FT08].

Lemma 20. *For any $\beta < j(\kappa)$ and any subset S of $j(F(\kappa))$ of size at most $j(\kappa)$ in $M[j(G_\kappa)]$, the sequence $\langle t_\xi^0 \restriction \beta \mid \xi \in S \rangle$ belongs to $M[j(G_\kappa)]$.*

Proof. Write $\beta = j(f_0)(a)$ where $f_0 : V_\kappa \rightarrow \kappa$ and $a \in V_\theta$. Let $C = \{\lambda < \kappa \mid f_0'' V_\lambda \subseteq \lambda \text{ and } \lambda \text{ is a limit cardinal}\}$. By Lemma 6 it follows that $S = j(f)(b)$ where $f : V_\kappa \rightarrow [F(\kappa)]^{\leq \kappa}$ and $b \in V_\theta$. Since $S \subseteq j(\bigcup \text{ran}(f))$ it can be assumed without loss of generality that $S = j(\bar{S})$ for some $\bar{S} \in [F(\kappa)]^{\leq \kappa}$. Let $\langle \bar{\alpha}_i \mid i < \kappa \rangle$ be an enumeration of \bar{S} . Then $j(\langle \bar{\alpha}_i \mid i < \kappa \rangle) = \langle \alpha_i \mid i < j(\kappa) \rangle$ is an enumeration of S . One can easily see that

$$D = \{\bar{p} \in \text{Sacks}(\kappa, F(\kappa)) \mid \text{for each } i < \kappa, C(\bar{p}(\bar{\alpha}_i)) \subseteq C \setminus (i+1)\}$$

is a dense subset of $\text{Sacks}(\kappa, F(\kappa))$. Let $\bar{p} \in H_\kappa \cap D$. Then for each $i < j(\kappa)$, $C(j(\bar{p})(\alpha_i)) \subseteq C \setminus (i+1)$. Thus, for each α_i , the tree $j(\bar{p})(\alpha_i)$ has no splits between κ and α . If $\kappa \leq i < j(\kappa)$ then $j(\bar{p})(\alpha_i)$ does not split between 0 and α . If $\kappa \leq i < j(\kappa)$ then $t_{\alpha_i}^0 \restriction \alpha$ is the unique element of $j(\bar{p})(\alpha_i)$ of length α . If $i < \kappa$, then $t_{\alpha_i}^0 \restriction \alpha$ is the unique element of $j(\bar{p})(\alpha_i)$ that extends $t_{\alpha_i}^0 \restriction \kappa$ and takes on value 0 at κ . \square

By Lemma 20, \tilde{p} is in $M[j(G_\kappa)]$ and is thus a condition in $j(\mathbb{R}_\kappa)$. Furthermore, \tilde{p} meets D and since $t_\xi^0 \subseteq \tilde{p}(\xi)$ for each $\xi < F(\kappa)$, it follows that \tilde{p} is in g . This establishes that g is $M[j(G_\kappa)]$ -generic for $j(\mathbb{R}_\kappa)$. Thus the embedding lifts to $j : V[G_\kappa][H_\kappa] \rightarrow M[j(G_\kappa)][j(H_\kappa)]$.

3.3. Lifting j Through $\mathbb{Q}_{[\kappa^+, \bar{\kappa}]} * \mathbb{P}_{[\bar{\kappa}, \delta]}$. By Lemma 14, the poset $\mathbb{Q}_{[\kappa^+, \bar{\kappa}]} * \mathbb{P}_{[\bar{\kappa}, \delta]}$ is $\leq \kappa$ -distributive in $V[G_\kappa][H_\kappa]$. Thus, from Lemma 7 one sees that $j'' H_{[\kappa^+, \bar{\kappa}]} * G_{[\bar{\kappa}, \delta]}$ generates an $M[j(G_\kappa)]$ $[j(H_\kappa)]$ -generic filter for $j(\mathbb{Q}_{[\kappa^+, \bar{\kappa}]} * \mathbb{P}_{[\bar{\kappa}, \delta]})$, call it $j(H_{[\kappa^+, \bar{\kappa}]} * G_{[\bar{\kappa}, \delta]})$. Thus j lifts to $j : V[G_\delta] \rightarrow M[j(G_\delta)]$ where $j(G_\delta) := j(G_\kappa) * (j(H_\kappa) \times j(H_{[\kappa^+, \bar{\kappa}]}) * j(G_{[\bar{\kappa}, \delta]}))$.

3.4. Verifying strongness for A . Let me argue that the lifted embedding $j : V[G_\delta] \rightarrow M[j(G_\delta)]$ satisfies $j(A) \cap \mu = A \cap \mu$. This will follow from the next fact.

Fact 21.

- (1) $j(\dot{A}) \cap V_\theta = \dot{A} \cap V_\theta$
- (2) $j(G_\delta) = G_{\gamma_0} * \tilde{G}_{[\gamma_0, \theta]} * \tilde{G}_{[\theta, j(\kappa)]}$ agrees with G_δ up to μ' since $\mu' < \gamma_0$.
- (3) $j(u) \restriction \mu' = u \restriction \mu'$

Using the above fact, one has the following.

$$\begin{aligned}
 A \cap \mu &= \dot{A}^{G_\delta} \cap \mu \\
 &= (\dot{A} \cap V_{\mu'})^{G_{\mu'}} \cap \mu && \text{(using the definition of } u) \\
 &= (j(\dot{A}) \cap V_{\mu'})^{G_{\mu'}} \cap \mu && \text{(by Fact 21(1))} \\
 &= j(\dot{A})^{j(G_\delta)} \cap \mu && \text{(by Fact 21(2) and (3))} \\
 &= j(A) \cap \mu
 \end{aligned}$$

This completes the proof of Theorem 1. \square

4. CONCLUSION AND SOME OPEN QUESTIONS

Cummings and Shelah [CS95] generalized Easton's Theorem [Eas70], by forcing to control not only the continuum function $\kappa \mapsto 2^\kappa$ on the regular cardinals, but to control the bounding number $\mathfrak{b}(\kappa)$ and dominating number $\mathfrak{d}(\kappa)$ on regular cardinals. Thus a natural extension of Question 1 is: To what extent can one control the function $\kappa \mapsto (\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^\kappa)$ on the regular cardinals by forcing while preserving large cardinals? In particular, to what extent can one control the behavior of $\kappa \mapsto (\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^\kappa)$ on the regular cardinals while preserving a Woodin cardinal?

Recall that δ is a *Shelah cardinal* if for every function $f : \delta \rightarrow \delta$ there is a transitive class N and an elementary embedding $j : V \rightarrow N$ with critical point δ such that $V_{j(f)(\delta)} \subseteq N$. Since every Shelah cardinal is measurable (and more), it follows that under the assumption that δ is

a Shelah cardinal, the continuum function has less freedom than under the assumption that δ is a Woodin cardinal. If δ is a Shelah cardinal, which Easton functions can one force to agree with the continuum function while preserving the Shelahness of δ ?

REFERENCES

- [Cod12] Brent Cody. *Some results on large cardinals and the continuum function*. PhD thesis, The Graduate Center of the City University of New York, 2012.
- [CS95] James Cummings and Saharon Shelah. Cardinal invariants above the continuum. *Annals of Pure and Applied Logic*, 75(3):251–268, 1995.
- [Cum92] James Cummings. A model in which GCH holds at successors but fails at limits. *Transactions of the American Mathematical Society*, 392(1):1–39, 1992.
- [Cum10] James Cummings. Iterated forcing and elementary embeddings. In Akihiro Kanamori and Matthew Foreman, editors, *Handbook of Set Theory*, volume 2, chapter 14, pages 775–883. Springer, 2010.
- [DF08] Natasha Dobrinen and Sy David Friedman. Homogeneous iteration and measure one covering relative to HOD. *Archive of Mathematical Logic*, 47(7–8):711–718, 2008.
- [Eas70] William B. Easton. Powers of regular cardinals. *Annals of Mathematical Logic*, 1:139–178, 1970.
- [FH08] Sy David Friedman and Radek Honzik. Easton’s theorem and large cardinals. *Annals of Pure and Applied Logic*, 154(3):191–208, 2008.
- [FT08] Sy David Friedman and Katherine Thompson. Perfect trees and elementary embeddings. *Journal of Symbolic Logic*, 73(3):906–918, 2008.
- [Ham97] Joel David Hamkins. Canonical seeds and prikry trees. *The Journal of Symbolic Logic*, 62(2):373–396, 1997.
- [HW00] Joel David Hamkins and W. Hugh Woodin. Small forcing creates neither strong nor woodin cardinals. *Proceedings of the American Mathematical Society*, 128(10):3025–3029, 2000.
- [Jec03] Thomas Jech. *Set Theory: The Third Millennium Edition, revised and expanded*. Springer, 2003.
- [Kan80] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Annals of Mathematical Logic*, 19(1–2):97–114, 1980.
- [Kan03] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer, second edition, 2003.
- [Men76] Telis K. Menas. Consistency results concerning supercompactness. *Transactions of the American Mathematical Society*, 223:61–91, 1976.

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